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NOTE

JOINT EXTENSION OF TWO THEOREMS OF KOTZIG ON 3-POLYTOPES

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The weight of an edge in a graph is the sum of the degrees of its end-vertices. It is proved that in each 3-polytope there exists either an edge of weight at most 13 for which both incident faces are triangles, or an edge of weight at most 10 which is incident with a triangle, or else an edge of weight at most 8. All the bounds 13, 10, and 8 are sharp and attained independently of each other.

The *weight* of an edge in a graph is defined to be the sum of the degrees of its end-vertices; the weight, w, of a graph is the minimal weight of its edges. Kotzig proved [5] that $w \leq 13$ for each 3-polytope and [6] $w \leq 8$ for each bipartite 3-polytope. Both bounds 13 and 8 are sharp as shown by the duals of the (3,10,10) Archimedian solid and the (3,5,3,5) Archimedian solid.

An edge is called *semiweak* if it is incident with at least one triangular face; the *semiweak weight*, w^*, of a graph is the minimal weight of its semiweak edges. Similarly, an edge is *weak* if both the edges incident with it are triangles; the *weak weight* is denoted by w^{**}. Clearly, $w^{**} \geq w^* \geq w$.

Recall that the *normal* planar maps are defined to contain no vertices or faces incident with less then three edges. Trivial examples show that if a planar map fails to be normal, then w may be arbitrarily large. Recall also that due to Steinitz Theorem [8] (for more accessible reference see [4, p.452]), 3-polytopes are distinguished among all normal maps by the property that their graphs are 3-connected.

In [3], I proved that $w \leq 13$ for each normal planar map, thus confirming, in particular, the same conjecture raised by Erdős for planar graphs without loops or multiple edges and of minimal degree three and established by Barnett (see [4, pp.454 and 467]). Moreover, for each normal planar map one has either $w^{**} \leq 13$, or $w^* \leq 12$, or $w \leq 11$, see [2]. These and similar structural results were discovered as a tool for solving some coloring problems on planar graphs [2]. For example, Kronk and Mitchem's conjecture [5], $\chi_{\text{vef}} \leq \Delta + 4$, on the entire coloration of the vertices, V, edges, E, and faces, F, of a planar graph having the maximal degree Δ was confirmed [1] for all $\Delta \geq 10$. This had been confirmed earlier for $\Delta = 3$ only [5]. Moreover, for $\Delta \geq 14$, I proved [1] that $\chi_{\text{vef}} \leq \Delta + 2$, which is already the best possible as shown by the star graph $K_{1,\Delta}$ (there are $\Delta + 2$ pairwise adjacent or

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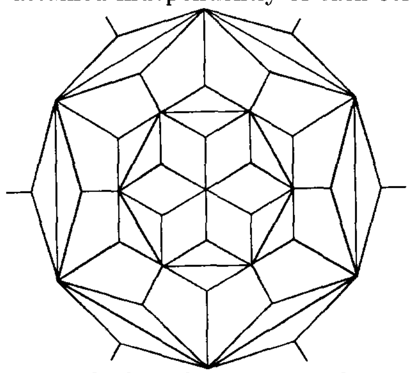
incident elements in it: the central vertex, all Δ edges, and the unique face, so that $\Delta+2$ colors are needed to color them).

The purpose of the present note is to complete these structural results by the final

Theorem. *For each normal planar map, either* $w^{**} \leq 13$, *or* $w^{*} \leq 10$, *or else* $w \leq 8$; *moreover, all the bounds* 13, 10, *and* 8 *are sharp.*

Proof. The figure below represents a graph with $w^{**}=14$, $w^{*}=10$, and $w=9$ (the six outgoing edges meet at the South Pole if the central vertex corresponds to the North Pole); the two Archimedian constructions mentioned above have $w^{**}=w^{*}=w=13$ and $w^{**}=w^{*}=\infty$, $w=8$, respectively. It follows that the bounds in our Theorem are sharp and attained independently of each other.



Let G be a counterexample for Theorem, i.e., have $w^{**} \geq 14$, $w^{*} \geq 11$, and $w \geq 9$. The Euler formula $|V|-|E|+|F|=2$ for G may be rewritten using the relations

$$2|E| = \sum_{v\in V} s(v) = \sum_{f\in F} r(f) = \sum_{i\geq 3} i|F_i|$$

as

$$\sum_{v\in V}(s(v)-6) + \sum_{i\geq 4}(2i-6)+|F_i| = -12, \tag{1}$$

where $s(v)$ is the degrees of a vertex v, $r(f)$ is the size of a face f, and F_i is the set of i-faces. Vertices and faces incident precisely with i edges, are called i-vertices and i-faces, respectively.

We attribute to each vertex, v, a *charge* $g(v)=s(v)-6$ and to each face, f, a *charge* $g(f)=2r(f)-6$. Now (1) transforms into

$$\sum_{v\in V} g(v) + \sum_{f\in F} g(f) = -12. \tag{2}$$

These charges will now be locally redistributed, keeping their sum constant, according to the following Rules 1 and 2 (we define an edge to be *strictly semiweak* if it is semiweak but not weak):

Rule 1. *Each non-triangular face, f, transfers to each vertex, v, in its boundary:*
1 *if* $s(v)=3$;
1/2 *if* $4 \le s(v) \le 5$,
or if $f=[\dots uvw \dots]$ *with* $s(u)\ge 7$, $s(v)\ge 7$, $s(w)\ge 5$.

Rule 2. *Let* $e=(w,v)$ *be an edge with* $s(w)\ge 7$, $s(v)\le 5$. *Then the following charge is transferred along* e *from* w *to* v:
1 *if* $s(v)=3$ *and* e *is weak;*
1/2 *if* $s(v)=3$ *and* e *is strictly semiweak,*
or if $s(v)=4$ *and* e *is weak;*
1/4 *if* $s(v)=4$ *and* e *is strictly semiweak;*
1/5 *if* $s(v)=5$ *and* e *is weak.*

New charges of the vertices and faces of G are denoted by a function h. Then

$$\sum_{v\in V} h(v) + \sum_{f\in F} h(f) = -12. \tag{3}$$

The remainder of our proof consists in verifying $h(x)\ge 0$ for each $x\in V\cup F$; this will yield an obvious contradiction with (3). Of course, Rules 1 and 2, as well as our subsequent arguments, entirely depend on the properties $w^{**}\ge 14$, $w^{*}\ge 11$, and $w\ge 9$ of G.

Let f be a k-face. If $k=3$ then $h(f)=g(f)=0$. Let $k\ge 4$ and $f=[\dots v_k v_1 v_2 v_3 \dots]$. Observe that since $w\ge 9$, the vertices which receive 1 from f each are not consecutive in the boundary of f. If $k=4$, we have $g(f)=2$. At most two of the v_i may receive 1 from f each; if, say, v_1 and v_3 are such, then it is clearly seen from Rule 1 that v_2 and v_4 do not receive anything from f, and $h(f)\ge 2-2\cdot 1=0$. If precisely one of v_i, say v_1 has degree 3, then at most two of v_2, v_3, v_4 may receive 1/2 each, and $h(f)\ge 2-1-2\cdot 1/2=0$. Finally, if $s(v_i)\ge 4$ for all $i\ge 4$, then $h(f)\ge 2-4\cdot 1/2=0$. If $k=5$, we have $g(f)=4$, but since at most two of the v_i have degree 3, we have $h(f)\ge 4-2\cdot 1-3\cdot 1/2>0$. To complete the analysis, observe that if $k\ge 6$ then trivially $h(f)\ge g(f)-k=2k-6-k\ge 0$.

Let $v\in V$. It is a direct consequence of Rule 2 that $h(v)=0$ for all vertices of degree 3 or 4. The same is true for those 5-vertices which are not incident with non-triangular faces. For all other 5-vertices, it is easily verified that $h(v)>0$.

Case 1: $s(v)=6$. Since v does not transfer anything due to the properties $w^{**}\ge 14$ and $w^{*}\ge 11$ and Rule 2, one has $h(v)=g(v)=0$.

Case 2: $s(v)=7$. The transfer from v is possible only to 4-vertices, and only along strictly semiweak edges. But no two such edges-conductors are consecutive since $w^{*}\ge 11$. It follows from a small case analysis that the number of such edges is at most four; on the other hand, each of them conducts 1/4; therefore $h(v)\ge 1-4\cdot 1/4=0$.

Case 3: $s(v)=8$. The transfer is now possible along strictly semiweak edges only, i.e., it does not exceed 1/2 for each edge. But $g(v)=2$, so we should only investigate the possibility that the number of edges-conductors is greater than four. A small case analysis however shows that the number of conductors is never greater than five. Moreover, when the number of such edges is five, the following situation takes place: The neighbours of v may be denoted in a clockwise order as $v_1, v_2, \dots, v_8$

so that v_1, v_3, v_4, v_6, and v_7 (or v_8 instead of v_7, which yields a symmetrical picture) have degree 3; furthermore, the only non-triangular faces incident with v are $[\ldots v_3vv_4\ldots]$, $[\ldots v_6vv_7\ldots]$ and $[\ldots v_8vv_1\ldots]$. It remains to observe that in this situation, v receives 1/2 from $[\ldots v_8vv_7\ldots]$ due to the last part of Rule 1, so that $h(v)=8-6+1/2-5\cdot 1/2=0$.

Case 4: $9 \leq s(v) \leq 10$. In what follows, we make use of the observation that no weak edge-conductor is a neighbour of any edge-conductor (since $w^* > 10$, while the transfer occurs to vertices of degree at most five only). Note that since $w^{**} \geq 14$, v is not incident with those weak edges which end in 3-vertices, i.e., every edge transfers from v at most 1/2. To simplify the end of the proof, we employ the following averaging argument. Let us take 1/6 off each edge that transfers 1/2 from v and direct this charge along such a neighbour edge of e in the boundary of f (see Rule 2) which is not a conductor. (At least one such neighbour does exist since e is semiweak and $w^* > 10$.) Now every edge transfers from v at most $1/2-1/6=1/6+1/6=1/3$, therefore $h(v)\geq s(v)-6-s(v)\cdot 1/3\geq 0$.

Case 5: $s(v)\geq 11$. Now we similarly twice take 1/4 off every edge, e, which transfers 1 from v and direct these charges along the two neighbour edges of e. After that, each edge transfers from v at most $1-2\cdot 1/4=2\cdot 1/4=1/2$; therefore if $s(v)\geq 12$, we already have what we need: $h(v)\geq s(v)-6-s(v)\cdot 1/2\geq 0$. Consider the last subcase $s(v)=11$. Observe that each strictly semiweak edge-conductor, e, is a neighbour of some non-conductor, e', and will therefore after averaging transfer from v at most 1/4. It follows, if there are at least two strictly semiweak conductors at v, then $h(v) \geq 11-5-9\cdot 1/2-2/4=0$. But if v is incident with precisely one strictly semiweak conductor, then it is incident with at most four weak conductors, that yields $h(v)\geq 11-6-4\cdot 1-1/2>0$. Finally, if v is not incident with strictly semiweak conductors, then it suffices to note that v is incident with at most five conductors at all due to the parity reasons, and $h(v)\geq 11-6-5\cdot 1=0$.

Thus, we have verified that $h(x)\geq 0$ for each $x\in V\cup F$, which violates (3). ∎

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